

# Observable effects in a class of spherically symmetric static Finsler spacetimes

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## Abstract

After some introductory discussion of the definition of Finsler spacetimes and their symmetries, we consider a class of spherically symmetric and static Finsler spacetimes which are small perturbations of the Schwarzschild spacetime. The deviations from the Schwarzschild spacetime are encoded in three perturbation functions  $\phi_0(r)$ ,  $\phi_1(r)$  and  $\phi_2(r)$  which have the following interpretations:  $\phi_0$  perturbs the time function,  $\phi_1$  perturbs the radial length measurement and  $\phi_2$  introduces a spatial anisotropy which is a genuine Finsler feature. We work out the equations of motion for freely falling particles and for light rays, i.e. the timelike and lightlike geodesics, in this class of spacetimes, and we discuss the bounds placed on the perturbation functions by observations in the Solar system.

## 1 Introduction

Since its discovery almost hundred years ago, general relativity has proven to give a very successful description of our universe. Nonetheless, there are good reasons for investigating gravitational theories that are more general than general relativity. There are many theoretical predictions, in particular from quantum gravity ideas, that general relativity should be replaced by a more general theory at some scale. In order to confront such theoretical predictions with experiments, it is necessary to theoretically study all observable effects of the more general theory. This will tell by what sort of future experiments deviations from general relativity could be observed, and to what accuracy general relativity is verified by present day observation. The PPN formalism provides a mathematical framework for doing so; however, it is restricted to metrical theories in the strict sense, i.e., to theories where the gravitational field is described by a pseudo-Riemannian metric tensor of Lorentzian signature, as in general relativity. For other theories, no such universal framework exists.

In this paper we want to investigate Finsler gravity theories, i.e., theories where the pseudo-Riemannian metric of general relativity is replaced with a Finsler metric. Finsler metrics are characterised by a Lagrangian function that is still homogeneous with respect to the velocities, but not necessarily given by a quadratic form. The most important feature that distinguishes a Finsler metric from a pseudo-Riemannian metric is in the fact that it breaks spatial isotropy even in “infinitesimally small regions”, i.e., mathematically speaking, on the tangent space. It is true that up to now there is no observational indication for such an anisotropy. (Note, however, Bogovslovsky’s [5] attempt to explain apparent violations of the GZK limit of cosmic rays as an effect of a spatial anisotropy.) At the position of the Earth, deviations from isotropy are strongly restricted by experiments of the Michelson-Morley type [14]. However, this result is based on the assumption that the armlength of the Michelson-Morley interferometer is to be determined not with the Finsler metric but with an independent Lorentzian background metric. If one assumes, by contrast, that the metric which determines the length of solid bodies shows the same sort of anisotropy as the metric that determines the light cones, then a Michelson-Morley-type experiment would give a null result.

Finsler manifolds have been considered as possible spacetime models by a large number of authors. A fairly complete list of the pre-1985 literature can be found in Asanov’s book [2]. There are several quite different motivations

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for considering Finsler manifolds as possible spacetime models. Apart from the aesthetic appeal Finsler geometry has for many authors, Finsler spacetimes have been recently suggested as a possible explanation for dark matter [6], and they have been used in an attempt for explaining the Pioneer anomaly [15]. (As it has now become clear that the latter can be explained as a thermal recoil effect [20], also cf. [8] and [28], this motivation should be considered as obsolete.) At a more fundamental level, it has been shown that Finsler geometry naturally comes up in models motivated by quantum gravity ideas [10], in particular in Very Special Relativity [9] and in other theories with violation of Lorentz invariance [13].

We take this as the motivation for investigating, in this article, the observational bounds on a spherically symmetric and static Finsler perturbation of the standard general relativity model of our Solar system. To that end we consider the effect such a Finsler perturbation would have on the motion of freely falling particles and light rays. In contrast to the above-mentioned Michelson-Morley-type experiments, we will not need any assumption on the behaviour of (“rigid”) extended bodies under the influence of a Finsler perturbation.

As the class of *all* spherically symmetric and static Finsler spacetimes is unmanageable (see Section 2 below), we need a special ansatz. As we want to discuss the motion of particles and of light rays, we need a Finsler spacetime in which both timelike and lightlike geodesics are well defined. This is an important issue, because in the literature one can find many Finsler spacetimes in which the notion of lightlike geodesics is *not* well defined. As this fact is glossed over in many articles, we discuss it in Section 2 below in some detail. Roughly speaking, three different definitions of Finsler spacetimes can be found in the literature: The one most frequently, though often implicitly, used in physics texts can be found in Asanov’s book [2]; an alternative one is due to Beem [4] and a quite recent one, which is a generalisation of Beem’s, is due to Pfeifer and Wohlfarth [17]. As we will outline in Section 2 below, none of them is appropriate for our purpose: Asanov’s definition does not allow to define lightlike geodesics, while Beem’s definition is slightly too restrictive to define staticity in the most convenient way; the latter observation is unaffected by Pfeifer and Wohlfarth’s generalisation. Therefore, we will introduce our own definition of Finsler spacetimes in Section 2 below which is a slight generalisation of Beem’s definition. On the basis of this definition, we will then consider in Section 3 a special class of Finsler spacetimes that are perturbations of the Schwarzschild metric. The perturbations preserve spherical symmetry and staticity. In this way we arrive at a formalism that allows us to quantitatively study, in Section 5 to 8, hypothetical Finsler deviations from general relativity in the Solar system, not only in terms of effects on particles but also on light rays; as in general relativity, our light rays are defined as geodesics whose initial vectors lie on a unique light cone that determines the causal structure of spacetime. We believe that such a formalism did not exist before. It is true that Roxburgh [21] set up a PPN formalism for Finsler gravity, cf. Roxburgh and Tavakol [22] for related material. This, however, was restricted to the very special case of a Finsler metric whose light cones coincide with the light cones of a pseudo-Riemannian metric; thereby any Finsler effect on the lightlike geodesics was excluded. There is also work by Aringazin and Asanov [1, 3] on Finsler generalisations of the Schwarzschild metric and possible observable effects. However, this is based on Asanov’s definition for which lightlike geodesics are not defined. Earlier work by Coley [7] is also (implicitly) based on Asanov’s definition. More recently, Pfeifer and Wohlfarth [18] have considered a certain Finsler perturbation of the linearised Schwarzschild metric; however, their ansatz is quite different from ours insofar as it introduces birefringence.

In analogy to the PPN formalism, our analysis will be purely kinematical, not using any field equation. Several attempts of establishing a Finsler generalisation of Einstein’s (vacuum) field equation have been brought forward, see Rund and Beare [24] (cf. Asanov [2], pp 110), Rutz [26], and Pfeifer and Wohlfarth [18]. However, it seems fair to say that no generally accepted Finsler version of a field equation exists so far.

As an aside, we mention that Finsler spacetimes in the sense considered in this paper provide a counter-example to the Schiff conjecture. In its original version, brought forward by L. Schiff in 1960 [27], this conjecture said that a theory must satisfy Einstein’s equivalence principle if it satisfies the weak equivalence principle. In our Finsler spacetimes there is a unique timelike geodesic for every timelike initial condition, so the weak equivalence principle is satisfied. However, as the theory is not based on a pseudo-Riemannian metric, Einstein’s equivalence principle is violated.

## 2 Definition of Finsler spacetimes and their symmetries

Historically, Finsler geometry was first established for positive definite metrics. In this case, which is covered in standard text-books such as Rund [23], a Finsler structure is defined in terms of a function  $F(x, \dot{x})$  that is positive and sufficiently smooth on the set of all tangent vectors  $(x, \dot{x})$  with  $\dot{x} \neq 0$ , and positively homogeneous of degree one, i.e.,  $F(x, k\dot{x}) = kF(x, \dot{x})$  for all  $k > 0$ . The Finsler metric is then introduced as the Hessian

$$g_{\mu\nu}(x, \dot{x}) = \frac{\partial^2 \mathcal{L}(x, \dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}, \quad (1)$$

where  $\mathcal{L}(x, \dot{x}) = F(x, \dot{x})^2$ , and it is required that this be positive definite for all  $\dot{x} \neq 0$ . The affinely parametrised Finsler geodesics are the solutions to the Euler-Lagrange equations of the Lagrangian  $\mathcal{L}(x, \dot{x})$ ,

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}. \quad (2)$$

For applications to spacetime theory one would like to have a Finsler metric of Lorentzian signature, and one would like to have timelike, lightlike and spacelike geodesics. This requires a modified definition of Finsler structures where the Lagrangian  $\mathcal{L}$  is no longer positive (hence not the square of a real-valued function  $F$ ) on the set of *all* non-zero tangent vectors. Therefore Asanov [2] defines a Finsler structure in terms of a function  $F(x, \dot{x})$  that has the same properties as in the positive definite formalism, but is given only on some subset of the tangent bundle which he calls the “admissible vectors”. The admissible vectors are to be interpreted as timelike. Typically, such a Finsler function  $F$  involves the square root of an expression that becomes negative on part of the tangent bundle; vectors where this happens are not admissible. In Asanov’s formalism, timelike geodesics are well-defined as the solutions to the Euler-Lagrange equations of the Lagrangian  $\mathcal{L} = F^2$  (or  $\mathcal{L} = -F^2$ , depending on the choice of signature) with admissible initial conditions. However, lightlike geodesics are not well-defined. What one would like to define as lightlike vectors are the ones on the boundary of the set of admissible vectors; there, however, the Euler-Lagrange equations break down because of zeros in the denominator. This formalism of Asanov, in which the Finsler structure is well-behaved only on the timelike vectors, is used in many physics papers on indefinite Finsler metrics, usually more implicitly than explicitly. The weakness of this approach is in the fact that there is no straightforward way of defining light rays in this setting. Asanov suggests a notion of light rays (see Chapter 7 in [2]) that depends on the choice of an auxiliary vector field. As the physical meaning of this auxiliary vector field is obscure, we do not think that this definition of light rays is satisfactory, from a physical point of view. Therefore, as we want to consider the equation of motion of light rays, we find Asanov’s definition of Finsler structures inappropriate for the purpose of this paper.

Fortunately, there is an alternative definition. Finsler metrics of Lorentzian signature were considered by Beem [4] in a way that is free from the above-mentioned drawbacks. In Beem’s formalism there is no analogue of the Finsler function  $F$ ; the Finsler structure is rather given directly in terms of the Lagrangian  $\mathcal{L}(x, \dot{x})$ , which should be sufficiently smooth (Beem requires it to be of class  $C^4$ ) and real-valued for all  $x$  and all  $\dot{x} \neq 0$ , it should be positively homogeneous of degree two,

$$\mathcal{L}(x, k\dot{x}) = k^2 \mathcal{L}(x, \dot{x}) \quad \text{for all } k > 0, \quad (3)$$

and the Finsler metric (1) should be non-degenerate with Lorentzian signature for all  $\dot{x} \neq 0$ . The non-degeneracy condition guarantees that the Euler-Lagrange equations (2) admit a unique solution to any initial condition  $(x(0), \dot{x}(0))$  with  $\dot{x}(0) \neq 0$ . These solutions are the affinely parametrised Finsler geodesics which are well-defined for timelike ( $\mathcal{L}(x, \dot{x}) < 0$ ), lightlike ( $\mathcal{L}(x, \dot{x}) = 0$ ) and spacelike ( $\mathcal{L}(x, \dot{x}) > 0$ ) tangent vectors  $\dot{x} \neq 0$ . So in Beem’s setting light rays can be unambiguously defined as lightlike geodesics, just as in standard general relativity. This is the reason why we consider, in this paper, a Finsler structure in the sense of Beem. Actually, for reasons that will become clear soon, we find it necessary to generalise Beem’s definition a little bit: Our Lagrangian  $\mathcal{L}$  will not be smooth (and not even  $C^2$ ) at all  $(x, \dot{x})$  with  $\dot{x} \neq 0$ ; the second derivative of  $\mathcal{L}$  will give undetermined expressions on a set of measure zero. However, there will still be a unique solution curve (geodesic) through each point  $(x, \dot{x})$  with  $\dot{x} \neq 0$ .

Another interesting generalisation of Beem’s definition was brought forward recently by Pfeifer and Wohlfarth [17, 18]. The main idea of their work is to allow for Lagrangians  $\mathcal{L}$  that are homogeneous of *any* degree. However, as they still assume  $\mathcal{L}$  to be “smooth” at all  $(x, \dot{x})$  with  $\dot{x} \neq 0$ , their generalisation is of no advantage for our purpose, although it might be fruitful for other applications.

Guided by Beem’s definition [4] we define a Finsler spacetime in the following way.

**Definition 1.** A Finsler spacetime is a 4-dimensional manifold  $M$  with a Lagrangian function  $\mathcal{L}$  that satisfies the following properties:

- (a)  $\mathcal{L}$  is a real-valued function on the tangent bundle  $TM$  minus the zero section, i.e.,  $\mathcal{L}(x, \dot{x})$  is defined for all  $(x, \dot{x})$  with  $\dot{x} \neq 0$ .
- (b)  $\mathcal{L}$  is positively homogeneous of degree two with respect to  $\dot{x}$ , i.e., eq. (3) holds.
- (c) The Finsler metric (1) is well-defined and has Lorentzian signature  $(-+++)$  for almost all  $(x, \dot{x})$  with  $\dot{x} \neq 0$ . (As usual, “almost all” means “up to a set of measure zero”.)
- (d) The Euler-Lagrange equations (2) admit a unique solution for every initial condition  $(x, \dot{x})$  with  $\dot{x} \neq 0$ ; at points where the Finsler metric is not well-defined this solution is to be constructed by continuous extension.

On a Finsler spacetime, we represent points in  $M$  by their coordinates  $x = (x^0, x^1, x^2, x^3)$  and points in the fibre  $T_x M$  of the tangent bundle by their induced coordinates  $\dot{x} = (\dot{x}^0, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ . We use Einstein’s summation convention for greek indices taking values 0,1,2,3 and for latin indices taking values 1,2,3.

Note that the homogeneity condition (b) of the Lagrangian implies that the Finsler metric is positively homogeneous of degree zero,

$$g_{\mu\nu}(x, k\dot{x}) = g_{\mu\nu}(x, \dot{x}) \quad \text{for all } k > 0, \quad (4)$$

and that the Lagrangian can be written in terms of the Finsler metric as

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} g_{\mu\nu}(x, \dot{x}) \dot{x}^\mu \dot{x}^\nu. \quad (5)$$

With the help of the Lagrangian we classify non-zero tangent vectors as timelike ( $\mathcal{L}(x, \dot{x}) < 0$ ), lightlike ( $\mathcal{L}(x, \dot{x}) = 0$ ) or spacelike ( $\mathcal{L}(x, \dot{x}) > 0$ ). We call the solutions to the Euler-Lagrange equations (2) the affinely parametrised Finsler *geodesics*. Again by the homogeneity condition (b) of the Lagrangian,  $\mathcal{L}(x, \dot{x})$  is a constant of motion; hence Finsler geodesics can be classified as timelike, lightlike or spacelike. We interpret the timelike geodesics as freely falling particles and the lightlike geodesics as light rays. This interpretation is in agreement with the idea that the (Finsler) spacetime geometry tells freely falling particles and light rays how to move, i.e., that no additional mathematical structures enter into the equations of motion for freely falling particles and light rays. We have already mentioned that some authors disagree with this hypothesis, as far as light rays are concerned. As the interpretation of lightlike Finsler geodesics as light rays is crucial for our work, some additional justification is given in the Appendix.

In this paper we want to consider a special class of Finsler spacetimes that will serve us as a model for the gravitational field around the Sun. We shall assume that this gravitational field is static and spherically symmetric. In order to make these notions precise we have to recall that symmetries of Finsler metrics are described in terms of (Finsler generalisations of) Killing vector fields. By definition, a vector field  $V = V^\mu \partial / \partial x^\mu$  on a Finsler spacetime  $M$  is a *Killing vector field* if and only if its flow, if lifted to  $TM$ , leaves the Lagrangian  $\mathcal{L}$  invariant. This condition can be rewritten in terms of the Finsler metric as

$$V^\mu \frac{\partial g_{\rho\sigma}}{\partial x^\mu} + \frac{\partial V^\tau}{\partial x^\nu} \dot{x}^\nu \frac{\partial g_{\rho\sigma}}{\partial \dot{x}^\tau} + \frac{\partial V^\tau}{\partial x^\rho} g_{\tau\sigma} + \frac{\partial V^\tau}{\partial x^\sigma} g_{\rho\tau} = 0. \quad (6)$$

Here the  $V^\mu$  depend on  $x$  only, whereas the  $g_{\mu\nu}$  depend on  $x$  and  $\dot{x}$ . The Finslerian Killing equation (6) is known since the early days of Finsler geometry, see Knebelman [12].

In the standard formalism of general relativity, one defines a spacetime as *stationary* if it admits a timelike Killing vector field  $V$  and as *static* if, in addition, this timelike Killing vector field  $V$  is orthogonal to hypersurfaces. If we exclude global pathologies (such as, e.g., the case that the quotient space  $M/V$  fails to be a Hausdorff manifold), the latter condition implies that the spacetime is a warped product of a 3-dimensional manifold with a (positive definite) Riemannian metric and the real line with a negative definite metric. We can use this property as the definition of staticity for Finsler spacetimes.

**Definition 2.** A Finsler spacetime  $(M, \mathcal{L})$  is *static* if  $M$  is diffeomorphic to a product,  $M \simeq \mathbb{R} \times N$ , and  $\mathcal{L}$  is of the form

$$\mathcal{L}(x^1, x^2, x^3, \dot{t}, \dot{x}^1, \dot{x}^2, \dot{x}^3) = \frac{1}{2} \left( g_{tt}(x^1, x^2, x^3) \dot{t}^2 + g_{ij}(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3) \dot{x}^i \dot{x}^j \right), \quad (7)$$

where  $t$  runs over  $\mathbb{R}$  and  $(x^1, x^2, x^3)$  are coordinates on  $N$ ; the temporal metric coefficient  $g_{tt}(x^1, x^2, x^3)$  must be negative and the spatial metric  $g_{ij}(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$  must be positive definite.

If the  $g_{ij}$  are independent of the  $\dot{x}^i$ , Definition 2 reduces to the definition of a static spacetime in the sense of general relativity. In any other case the limit of the  $g_{ij}$  for  $(\dot{x}^1, \dot{x}^2, \dot{x}^3) \rightarrow (0, 0, 0)$  depends on the direction in which this limit is performed; this follows immediately from eq. (4). As a consequence, the Finsler metric fails to be well-defined on vectors tangent to the  $t$ -lines. This is the reason why, in part (c) of Definition 1, the restriction to “almost all” non-zero tangent vectors was necessary to include proper Finsler Lagrangians of the form of eq. (7).

We now add the condition of spherical symmetry. By definition, a Finsler spacetime is spherically symmetric if it admits a 3-dimensional algebra of Killing vector fields that generate the rotation group  $\text{SO}(3)$  such that each of its orbits is diffeomorphic to the 2-sphere  $S^2$ . For a static Finsler spacetime as given in eq. (7), spherical symmetry means that we can choose the spatial coordinates as  $x^1 = r, x^2 = \vartheta, x^3 = \varphi$ , where  $r$  labels the group orbits and  $\vartheta$  and  $\varphi$  are standard coordinates on  $S^2$ , and that then  $g_{tt}$  depends on  $r$  only and the spatial part  $g_{ij}\dot{x}^i\dot{x}^j$  depends on  $r, \dot{r}$  and  $\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2$  only. For a derivation of the latter fact see McCarthy and Rutz [16, 25].

### 3 A class of spherically symmetric and static Finsler spacetimes

As the class of all spherically symmetric and static Finsler spacetimes is too big, we make a more special ansatz for our model of the Solar system. We assume that the Lagrangian  $\mathcal{L}$  is of the form

$$2\mathcal{L} = (h_{tt} + c^2\psi_0)\dot{t}^2 + \left( (h_{ij}h_{kl} + \psi_{ijkl})\dot{x}^i\dot{x}^j\dot{x}^k\dot{x}^l \right)^{\frac{1}{2}}. \quad (8)$$

Here

$$h_{tt}dt^2 + h_{ij}dx^i dx^j = h_{tt}dt^2 + h_{rr}dr^2 + r^2(\sin^2\vartheta d\varphi^2 + d\vartheta^2) \quad (9)$$

is a spherically symmetric and static Lorentzian metric. In this section and in the following one,  $h_{tt}$  and  $h_{rr}$  are arbitrary functions of  $r$ , but later they will be specified to be the Schwarzschild metric coefficients,

$$h_{tt} = -\frac{c^2}{h_{rr}} = -c^2 \left( 1 - \frac{2GM}{c^2 r} \right), \quad (10)$$

where  $c$  is the speed of light,  $G$  is the gravitational constant, and  $M$  is the mass of the gravitating body. The spatial perturbation  $\psi_{ijkl}$  is spherically symmetric and independent of  $t$ ,

$$\psi_{ijkl}\dot{x}^i\dot{x}^j\dot{x}^k\dot{x}^l = \psi_1(r)\dot{r}^4 + \psi_2(r)r^2\dot{r}^2(\sin^2\vartheta\dot{\varphi}^2 + \dot{\vartheta}^2) + \psi_3(r)r^4(\sin^2\vartheta\dot{\varphi}^2 + \dot{\vartheta}^2)^2 \quad (11)$$

and the time perturbation  $\psi_0$  is a function of  $r$  only.

Actually, ansatz (8) is less special than it might appear. The fourth-order term  $\psi_{ijkl}\dot{x}^i\dot{x}^j\dot{x}^k\dot{x}^l$  can be viewed as the leading order term in a general Finsler power-law perturbation of the spatial part of the metric. (We do not want

to consider a third-order term because it would violate the symmetry under spatial inversions  $\dot{x}^i \rightarrow -\dot{x}^i$ .) For this reason, we consider the Lagrangian (8) as a natural choice for our purpose.

In the following we refer to the dimensionless quantities  $\psi_A(r)$  as to the “perturbation functions”,  $A = 0, 1, 2, 3$ . Throughout this paper, we assume that the perturbation functions depend differentiably on  $r$  and are so small that we may linearise all equations with respect to the  $\psi_A(r)$  and their derivatives  $\psi'_A(r)$ . Differentiability and smallness of the  $\psi_A(r)$  guarantee that the Lagrangian  $\mathcal{L}(x, \dot{x})$  is real-valued, and the Finsler metric (1) is non-degenerate with Lorentzian signature for *almost* all  $(x, \dot{x})$  with  $\dot{x} \neq 0$ . The only points where this condition is violated are the points where the *spatial* velocity components are all zero,  $(\dot{x}^1, \dot{x}^2, \dot{x}^3) = (0, 0, 0)$ , but  $\dot{t} \neq 0$ . At these points, the Finsler metric gives undetermined expressions. We will see in the next section that, even through these points, the solutions to the Euler-Lagrange equations are uniquely determined by continuous extension, i.e., that our ansatz gives indeed a Finsler spacetime in the sense of Definition 1.

We are still free to transform the radial coordinate. We can remove this freedom, thereby reducing the number of perturbation functions from four to three. In the unperturbed (Schwarzschild) spacetime,  $r$  is an “area coordinate”, i.e., the area of the sphere at  $r$  is given by  $4\pi r^2$ . We can fix the radial coordinate by requiring that  $r$  has the same geometric meaning in the perturbed spacetime. From equations (8) and (11) we read that, in the perturbed spacetime, the sphere at  $r$  has area  $4\pi r^2(1 + \psi_3)$ . Hence, the desired condition is satisfied if we allow only perturbations with  $\psi_3 = 0$ . We are then left with three perturbation functions  $\psi_0$ ,  $\psi_1$  and  $\psi_2$ , and the Lagrangian (8) reads

$$2\mathcal{L} = (h_{tt} + c^2\psi_0)\dot{t}^2 + \left(\sqrt{h_{rr}^2 + \psi_1}\dot{r}^2 + r^2(\sin^2\vartheta\dot{\varphi}^2 + \dot{\vartheta}^2)\right) \sqrt{1 + \frac{(2h_{rr} - 2\sqrt{h_{rr}^2 + \psi_1} + \psi_2)r^2\dot{r}^2(\sin^2\vartheta\dot{\varphi}^2 + \dot{\vartheta}^2)}{\left(\sqrt{h_{rr}^2 + \psi_1}\dot{r}^2 + r^2(\sin^2\vartheta\dot{\varphi}^2 + \dot{\vartheta}^2)\right)^2}}. \quad (12)$$

From this expression we read that  $\mathcal{L}$  is the Lagrangian of a pseudo-Riemannian metric if and only if

$$2h_{rr} - 2\sqrt{h_{rr}^2 + \psi_1} + \psi_2 = 0. \quad (13)$$

In this case the equations of motion can be investigated in terms of the standard PPN formalism. If (13) does not hold, we have a proper Finsler geometry and the PPN formalism does not apply. We might say that the left-hand side of eq. (13) measures the “Finslerity” of our perturbed spacetime.

## 4 Equations of motion

We now discuss the solutions to the Euler-Lagrange equations (2) in our class of spherically symmetric and static Finsler spacetimes, i.e., the affinely parametrised Finsler geodesics. We restrict to timelike ( $\mathcal{L} < 0$ ) and lightlike ( $\mathcal{L} = 0$ ) geodesics, which are to be interpreted as freely falling particles and as light rays, respectively. For timelike geodesics we can fix the parametrisation by requiring  $2\mathcal{L} = -c^2$ ; then the affine parameter is equal to Finsler proper time  $\tau$ .

By symmetry, it suffices to consider particles and light rays in the equatorial plane  $\vartheta = \pi/2$ . Then the linearised version of the Lagrangian (12) reads

$$2\mathcal{L} = (1 + \phi_0)h_{tt}\dot{t}^2 + (1 + \phi_1)h_{rr}\dot{r}^2 + r^2\dot{\varphi}^2 + \frac{\phi_2 h_{rr} r^2 \dot{r}^2 \dot{\varphi}^2}{h_{rr}\dot{r}^2 + r^2\dot{\varphi}^2}. \quad (14)$$

Here we have introduced, for notational convenience, modified perturbation functions

$$\phi_0 = \frac{c^2\psi_0}{h_{tt}}, \quad \phi_1 = \frac{\psi_1}{2h_{rr}}, \quad \phi_2 = \frac{\psi_2 h_{rr} - \psi_1}{2h_{rr}^2}. \quad (15)$$

In terms of these modified perturbation functions, and after linearisation, the “non-Finsler condition” (13) simply reads  $\phi_2 = 0$ . Hence, in the linearised setting the “Finslerity” of our perturbed spacetime is measured just by  $\phi_2$ .

By equation (14), each of the perturbation functions  $\phi_0$ ,  $\phi_1$  and  $\phi_2$  has an obvious interpretation:  $\phi_0$  perturbs the time function  $t$ ,  $\phi_1$  perturbs the radial length measurement and  $\phi_2$  introduces a spatial anisotropy which is a genuine Finsler feature. Circular motion ( $\dot{r} = 0$ ) feels only  $\phi_0$  while radial motion ( $\dot{\varphi} = 0$ ) feels  $\phi_0$  and  $\phi_1$ ; the “Finslerity”  $\phi_2$  is felt only by motion that is neither circular nor radial.

Equation (14) is the form of the Lagrangian on which all our following results are based. We will now derive the equations of motion.

In addition to the constant of motion

$$\mathcal{L} = -\frac{c^2}{2} \text{ for freely falling particles} \quad (16)$$

or

$$\mathcal{L} = 0 \text{ for light,} \quad (17)$$

the  $t$  and  $\varphi$  components of the Euler-Lagrange equations give two more constants of motion  $E$  and  $L$ ,

$$-E = \frac{\partial \mathcal{L}}{\partial \dot{t}} = (1 + \phi_0) h_{tt} \dot{t}, \quad (18)$$

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2 \dot{\varphi} \left( 1 + \frac{\phi_2 h_{rr}^2 \dot{r}^4}{(h_{rr} \dot{r}^2 + r^2 \dot{\varphi}^2)^2} \right). \quad (19)$$

The three constants of motion  $\mathcal{L}$ ,  $E$  and  $L$  give us three equations that determine the geodesics. From these three equations we read that, by continuity, there is a unique geodesic even for initial conditions  $\dot{r}(0) = 0$ ,  $\dot{\varphi}(0) = 0$  and  $\dot{t}(0) \neq 0$ , for which the Euler-Lagrange equations yield undetermined expressions, namely a curve with  $\varphi = \text{constant}$ . This completes the proof that our Lagrangian defines a Finsler spacetime in the sense of Definition 1.

To within our linear approximation, the three conservation equations (14), (18) and (19) can be solved for  $\dot{t}$ ,  $\dot{\varphi}$  and  $\dot{r}^2$ ,

$$\dot{t} = \frac{-E}{h_{tt}} (1 - \phi_0), \quad (20)$$

$$\dot{\varphi} = \frac{L}{r^2} \left( 1 - \frac{\phi_2 \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{L^2}{r^2} \right)^2}{\left( 2\mathcal{L} - \frac{E^2}{h_{tt}} \right)^2} \right), \quad (21)$$

$$\dot{r}^2 = \frac{1}{h_{rr}} \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{L^2}{r^2} \right) \left( 1 - \phi_1 + \frac{\phi_2 L^2 \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{2L^2}{r^2} \right)}{r^2 \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} \right)^2} \right) + \frac{\phi_0 E^2}{h_{tt} h_{rr}}. \quad (22)$$

From these three equations we find

$$\frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}} = \frac{L h_{tt}}{-E r^2} \left( 1 + \phi_0 - \frac{\phi_2 \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{L^2}{r^2} \right)^2}{\left( 2\mathcal{L} - \frac{E^2}{h_{tt}} \right)^2} \right), \quad (23)$$

$$\left( \frac{dr}{dt} \right)^2 = \frac{\dot{r}^2}{\dot{t}^2} = \frac{h_{tt}^2}{E^2 h_{rr}} \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{L^2}{r^2} \right) \left( 1 + \phi_0 - \phi_1 + \frac{\phi_2 L^2 \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{2L^2}{r^2} \right)}{r^2 \left( 2\mathcal{L} - \frac{E^2}{h_{tt}} \right)^2} \right) + \frac{\phi_0 h_{tt}^2 \left( 2\mathcal{L} - \frac{L^2}{r^2} \right)}{E^2 h_{rr}}. \quad (24)$$

Equations (23) and (24) determine the trajectories if parametrised by coordinate time  $t$ . If we are interested only in the geometrical shape of the trajectory, but not in its parametrisation, we may use the equation

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{\dot{r}^2}{\dot{\varphi}^2} = \frac{r^4 \left(2\mathcal{L} - \frac{E^2}{h_{tt}} - \frac{L^2}{r^2}\right)}{h_{rr} L^2} \left(1 - \phi_1 + \frac{\phi_2 \left(4\mathcal{L} - \frac{2E^2}{h_{tt}} - \frac{3L^2}{r^2}\right)}{\left(2\mathcal{L} - \frac{E^2}{h_{tt}}\right)}\right) + \frac{\phi_0 E^2 r^4}{h_{rr} h_{tt} L^2}. \quad (25)$$

## 5 Circular orbits

For a particle ( $2\mathcal{L} = -c^2$ ) on a circular orbit, the equations  $dr/d\varphi = 0$  and  $d^2r/d\varphi^2 = 0$  must hold. By equation (25), these two conditions are equivalent to

$$-c^2 - \frac{E^2}{h_{tt}} - \frac{L^2}{r^2} + \frac{\phi_0 E^2}{h_{tt}} = 0, \quad (26)$$

$$\frac{E^2}{h_{tt}^2} h'_{tt} + \frac{2L^2}{r^3} + E^2 \left(\frac{\phi'_0}{h_{tt}} - \frac{\phi_0 h'_{tt}}{h_{tt}^2}\right) = 0. \quad (27)$$

With  $E^2$  and  $L^2$  determined this way, equation (23) yields

$$\left(\frac{d\varphi}{dt}\right)^2 = -\frac{h'_{tt}}{2r} \left\{1 + \frac{(\phi_0 h_{tt})'}{h'_{tt}}\right\}. \quad (28)$$

After inserting the Schwarzschild metric (10), we find

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{GM}{r^3} \left\{1 + \frac{c^2 r^2}{2GM} \left(\phi_0 \left(1 - \frac{2GM}{c^2 r}\right)\right)'\right\}. \quad (29)$$

If we denote the period by  $T$ , we have  $d\varphi/dt = 2\pi/T$ , and (29) gives a generalisation of the third Kepler law for circular orbits,

$$\frac{r^3}{T^2} \left\{1 - \frac{c^2 r^2}{2GM} \left(\phi_0 \left(1 - \frac{2GM}{c^2 r}\right)\right)'\right\} = \frac{GM}{4\pi^2}. \quad (30)$$

In the unperturbed Schwarzschild spacetime, the Kepler law

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2} \quad (31)$$

coincides with the Newtonian Kepler law, as is well known.

From an experimentalist's point of view, one may use the unperturbed Kepler law (31) as an operational definition of  $GM$ . According to standard general relativity this would lead to a constant value  $GM$ . With our perturbation, it would lead to an  $r$ -dependent value  $\widehat{GM}(r)$  that is related to the constant  $GM$  value from general relativity by

$$\widehat{GM}(r) = GM \left\{1 + \frac{c^2 r^2}{2GM} \left(\phi_0 \left(1 - \frac{2GM}{c^2 r}\right)\right)'\right\}. \quad (32)$$

We will now discuss the bounds imposed on  $\phi_0(r)$  by (32).

From observations, the gravitational constant  $G$  is known, at present, only up to a relative uncertainty of approximately  $10^{-4}$ . However, our knowledge of the product  $GM$ , where  $M$  denotes the Solar mass, is much better. The most recent value, taken from the webpage of the Jet Propulsion Laboratory <http://ssd.jpl.nasa.gov/?constants>, is

$$GM = 1.32712440018 \times 10^{20} \text{ m}^3 \text{ s}^{-2} \pm 8 \times 10^9 \text{ m}^3 \text{ s}^{-2}. \quad (33)$$



More specifically, we get a value for  $GM = 4\pi^2 a^3 / T^2$  from the observed values of the semi-major axis  $a$  and of the period  $T$  for each individual planet. As the periods are known with a higher accuracy than the semi-major axes we can write

$$\left| \frac{\widehat{GM}(a) - GM}{GM} \right| \lesssim 3\epsilon \quad (34)$$

where  $\epsilon$  is the accuracy with which the semi-major axis  $a$  is known. The values of  $\epsilon$  for the eight planets are shown in Table 1.

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
$a/(10^{10}\text{m})$	5.79	10.8	15.0	22.8	77.9	143	288	450
$\Delta a/\text{m}$	0.11	0.33	0.15	0.66	640	4200	$3.8 \times 10^4$	$4.8 \times 10^5$
$\epsilon$	$1.9 \times 10^{-11}$	$3.1 \times 10^{-11}$	$1.0 \times 10^{-11}$	$2.9 \times 10^{-11}$	$8.2 \times 10^{-9}$	$2.9 \times 10^{-8}$	$1.3 \times 10^{-7}$	$1.1 \times 10^{-6}$

Table 1: The first row of this table gives the semi-major axis  $a$  of each planet and the second row gives the formal standard deviation  $\Delta a$  of  $a$ . We have taken the values for  $\Delta a$  from Table 4 in Pitjeva [19] who reported on data determined by the Institute of Applied Astronomy of the Russian Academy of Sciences from observations made between 1913 and 2003. In the third row we calculated the accuracy  $\epsilon$  of the semi-major axis for each planet, assuming that the real error may be one order of magnitude bigger than the formal standard deviation,  $\epsilon = 10\Delta a/a$ .

In Section 8 below we will consider non-circular orbits. We will see that then  $\phi_1$  and  $\phi_2$  do have an effect. For the time being we will be satisfied with a rough order-of-magnitude estimate given by replacing each of the planets with a hypothetical planet that moves on a circular orbit with  $r = a$ . Then we can compare (32) with (34) and conclude that

$$\left| \left( \phi_0 \left( 1 - \frac{2GM}{c^2 r} \right) \right)' \right| \lesssim \frac{2GM}{c^2 r^2} 3\epsilon. \quad (35)$$

Integration from  $r_1$  to  $r_2$  yields

$$\left| \left( \phi_0(r_2) \left( 1 - \frac{2GM}{c^2 r_2} \right) - \phi_0(r_1) \left( 1 - \frac{2GM}{c^2 r_1} \right) \right) \right| \lesssim \frac{2GM}{c^2 r_1^2 r_2^2} |r_2 - r_1| 3\epsilon_{\max} \quad (36)$$

where  $\epsilon_{\max}$  is the maximal uncertainty between  $r_1$  and  $r_2$ . As  $2GM/(c^2 r)$  varies from  $10^{-8}$  near the Mercury orbit to  $10^{-10}$  near the Neptune orbit, (36) implies

$$r_1 \left| \frac{\phi_0(r_2) - \phi_0(r_1)}{r_2 - r_1} \right| \lesssim 10^{-16} \quad (37)$$

for all radii  $r_1$  and  $r_2$  between the Mercury orbit and the Neptune orbit. If both  $r_1$  and  $r_2$  are between the Mercury orbit and the Mars orbit, the bound is even three orders of magnitude smaller. In particular,

$$r |\phi_0'(r)| \lesssim 10^{-16} \quad (38)$$

between Mercury and Neptune, and even three orders of magnitude smaller between Mercury and Mars. Note that no assumption on monotonicity of the function  $\phi_0$  was needed for this result.

In analogy to the PPN formalism, one could assume that the perturbation functions are of the form

$$\phi_A(r) = \phi_{A1} \frac{2GM}{c^2 r} + O\left(\left(\frac{2GM}{c^2 r}\right)^2\right), \quad A = 0, 1, 2 \quad (39)$$

with constants  $\phi_{A1}$ . We have included the Schwarzschild radius  $2GM/c^2$  in this ansatz to make the  $\phi_{A1}$  dimensionless. Here we mean by  $GM$  the constant of Nature given by the fixed numerical value after the equality sign in (33), just as we mean by  $c$  the constant of Nature given by the numerical value 299 792 458 m/s. If we accept the ansatz (39) and if we assume that, as a reasonable first approximation, the terms of second and higher order can be neglected, the inequality (35) yields

$$|\phi_{01}| \left( 1 - \frac{4GM}{c^2 r} \right) \lesssim 3\epsilon. \quad (40)$$

With the best value of  $\epsilon$  taken from Table 1 we find

$$|\phi_{01}| \lesssim 3 \times 10^{-11}. \quad (41)$$

## 6 Radial free fall

In this section we consider a freely falling particle ( $2\mathcal{L} = -c^2$ ) that moves in the radial direction, i.e.  $\dot{\phi} = 0$ . By (19), the latter condition is equivalent to  $L = 0$ , so (24) simplifies to

$$\left( \frac{dr}{dt} \right)^2 = \frac{h_{tt}^2}{h_{rr} E^2} \left( -c^2 - \frac{E^2}{h_{tt}} \right) (1 + \phi_0 - \phi_1) - \frac{\phi_0 c^2 h_{tt}^2}{h_{rr} E^2}. \quad (42)$$

This expression gives the particle's velocity as it is measured by static observers with clocks that show coordinate time  $t$ . If we want to consider the same observers with clocks that show (Finsler) proper time  $\tau$ , we have to use the relation

$$c^2 d\tau^2 = -(1 + \phi_0) h_{tt} dt^2. \quad (43)$$

Then (42) can be rewritten as

$$\left( \frac{dr}{d\tau} \right)^2 = \frac{c^2 h_{tt}}{E^2 h_{rr}} \left\{ \left( c^2 + \frac{E^2}{h_{tt}} \right) (1 - \phi_1) + c^2 \phi_0 \right\}. \quad (44)$$

As a first application of this equation we want to discuss the acceleration of a particle from rest. From (44) we read that, at a point where  $dr/d\tau = 0$ , the equation

$$c^2 + \frac{E^2}{h_{tt}} + c^2 \phi_0 = 0 \quad (45)$$

must hold. By differentiating (44) with respect to  $\tau$ , and inserting (45) afterwards, we find the following expression for the acceleration from rest.

$$2 \frac{d^2 r}{d\tau^2} = - \frac{c^2 h'_{tt}}{h_{tt} h_{rr}} \left\{ 1 - \phi_1 + \phi'_0 \frac{h_{tt}}{h'_{tt}} \right\}. \quad (46)$$

With the Schwarzschild metric (10) this can be rewritten as

$$\frac{d^2 r}{d\tau^2} = - \frac{GM}{r^2} \left\{ 1 - \phi_1 - \phi'_0 r + \phi'_0 \frac{c^2 r^2}{2GM} \right\}. \quad (47)$$

In the unperturbed Schwarzschild spacetime (47) reduces to

$$\frac{d^2 r}{d\tau^2} = - \frac{GM}{r^2}. \quad (48)$$

As an alternative to the method discussed in Section 5, an experimentalist could use (48) as an operational definition of  $GM$ . According to standard general relativity, this would lead to the same constant value  $GM$  as the method of Section 5. With our perturbation, however, it would lead to an  $r$ -dependent value

$$\widetilde{GM}(r) = GM \left\{ 1 - \phi_1(r) - \phi'_0(r) r + \phi'_0(r) \frac{c^2 r^2}{2GM} \right\} \quad (49)$$

which is different from the  $\widehat{GM}(r)$  of (32),

$$\widehat{GM}(r) - \widetilde{GM}(r) = GM \{ \phi_0(r) + \phi_1(r) \}. \quad (50)$$

We see that the perturbation functions  $\phi_0$  and  $\phi_1$  can be determined, in principle, by observing circular orbits and radial acceleration from rest. With the bounds on  $\phi_0$  we have found from the observation of circular orbits in Section 5, we can now discuss the bounds on  $\phi_1$  that result from measurements of free-fall accelerations. Again, we are satisfied with a rough order-of-magnitude estimate. We can then say that, with the Pioneer anomaly explained as a thermal recoil effect, all observations of radial accelerations up to the Neptune orbit are in agreement with General Relativity to within the order of the anomalous Pioneer acceleration of  $9 \times 10^{-10} \text{m/s}^2$ . At the Neptune orbit,  $GM/r^2 \approx 6 \times 10^{-3} \text{m/s}^2$ . As a consequence, (49) suggests that

$$\left| \phi_1(r) - \frac{c^2 r}{GM} r \phi_0'(r) \left( 1 - \frac{2GM}{c^2 r} \right) \right| \lesssim \frac{9 \times 10^{-10}}{6 \times 10^{-3}}. \quad (51)$$

Using  $|a| - |b| \leq ||a| - |b|| \leq |a - b|$ , we find

$$|\phi_1(r)| \lesssim \left| \frac{c^2 r}{GM} r \phi_0'(r) \left( 1 - \frac{2GM}{c^2 r} \right) \right| + 1.5 \times 10^{-7}. \quad (52)$$

From Section 5 we know that  $\left| \frac{c^2 r}{2GM} r \phi_0'(r) \right| \lesssim 10^{-6}$  between the Mercury orbit and the Neptune orbit, hence

$$|\phi_1(r)| \lesssim 10^{-6} \quad (53)$$

for all  $r$  in this range.

If we assume that the perturbation functions have a fall-off behaviour according to (39), and if we neglect terms of second and higher order, (52) implies

$$|\phi_{11}| \frac{2GM}{c^2 r} \lesssim 2 |\phi_{01}| \left( 1 - \frac{2GM}{c^2 r} \right) + 1.5 \times 10^{-7}. \quad (54)$$

Evaluating at the Mercury orbit,  $\frac{2GM}{c^2 r} \approx 5 \times 10^{-8}$ , and using (41) yields

$$|\phi_{11}| \lesssim 3. \quad (55)$$

This is much less restrictive than the bound on  $\phi_{01}$  we had found before. Note, however, that a value of  $\phi_{11}$  of the order of unity gives only a small correction to  $h_{rr}$  of the Schwarzschild metric, because our ansatz (39) involves the Schwarzschild radius.

## 7 Effects on the paths of light rays

In this section we want to calculate the effect of our Finsler perturbation on the worldline of a light ray that comes in from a source at radial coordinate  $r_S$ , passes the Sun at a minimal value  $r_m$  of the radial coordinate, and goes out again to an observer at radial coordinate  $r_O$ .

To that end we have to evaluate (24) and (25) with the Schwarzschild metric coefficients (10) and  $\mathcal{L} = 0$ . For notational convenience, we will use the abbreviation

$$p(r) = r^{-2} \left( 1 - \frac{2GM}{c^2 r} \right) \quad (56)$$

throughout. With this abbreviation, and  $\mathcal{L} = 0$ , equation (25) can be rewritten in the following form.

$$\left(\frac{dr}{d\varphi}\right)^2 = r^4 \left(\frac{E^2}{c^2 L^2} - p\right) \left\{1 - \phi_1 + 2\phi_2 - 3\phi_2 c^2 p \frac{L^2}{E^2}\right\} - \frac{\phi_0 r^4 E^2}{c^2 L^2}. \quad (57)$$

We first determine how the constants of motion  $E$  and  $L$  depend on the minimum radius  $r_m$ . If we insert the value  $r = r_m$  into the right-hand side of (57), we must get zero. If we solve the resulting equation for  $E^2/L^2$ , we find

$$\frac{E^2}{L^2} = c^2 p(r_m) \left\{1 + \phi_0(r_m)\right\}. \quad (58)$$

Inserting this value into (57) yields

$$\left(\frac{dr}{d\varphi}\right)^2 = A(r) (1 - \alpha(r)) \quad (59)$$

where

$$A(r) = r^4 (p(r_m) - p(r)) \quad (60)$$

and

$$\alpha(r) = \phi_1(r) - \phi_2(r) \frac{2p(r_m) - 3p(r)}{p(r_m)} - (\phi_0(r_m) - \phi_0(r)) \frac{p(r_m)}{p(r_m) - p(r)}. \quad (61)$$

An analogous calculation puts (24) into the form

$$\left(\frac{dr}{dt}\right)^2 = B(r) (1 - \beta(r)) \quad (62)$$

where

$$B(r) = c^2 r^4 p(r)^2 \left(1 - \frac{p(r)}{p(r_m)}\right) \quad (63)$$

and

$$\beta(r) = \phi_1(r) - \phi_2(r) p(r) \frac{p(r_m) - 2p(r)}{p(r_m)^2} - \phi_0(r) - (\phi_0(r_m) - \phi_0(r)) \frac{p(r)}{p(r_m) - p(r)}. \quad (64)$$

Note that the perturbations  $\alpha(r)$  and  $\beta(r)$  depend not only on  $\phi_0(r)$ ,  $\phi_1(r)$  and  $\phi_2(r)$  but also on  $\phi_0(r_m)$ , because of (58).

## 7.1 Light deflection

From (59) we find

$$d\varphi = \left(1 + \frac{\alpha(r)}{2}\right) \frac{dr}{\sqrt{A(r)}} \quad (65)$$

and integration yields the deflection angle  $\Delta\varphi$ ,

$$\pi + \Delta\varphi = \left(\int_{r_m}^{r_S} + \int_{r_m}^{r_O}\right) \left(1 + \frac{\alpha(r)}{2}\right) \frac{dr}{\sqrt{A(r)}}. \quad (66)$$

If we denote by  $\Delta\varphi_0$  the deflection angle in the unperturbed Schwarzschild spacetime,

$$\pi + \Delta\varphi_0 = \left(\int_{r_m}^{r_S} + \int_{r_m}^{r_O}\right) \frac{dr}{\sqrt{A(r)}}, \quad (67)$$

the deflection angle in the perturbed spacetime reads

$$\Delta\varphi = \Delta\varphi_0 + \left(\int_{r_m}^{r_S} + \int_{r_m}^{r_O}\right) \frac{\alpha(r) dr}{2\sqrt{A(r)}}. \quad (68)$$

Using the mean value theorem, this result can be rewritten as

$$\Delta\varphi = \Delta\varphi_0 + \frac{\alpha(\tilde{r})}{2} (\pi + \Delta\varphi_0), \quad (69)$$

where  $\tilde{r}$  is some radius value between  $r_m$  and  $\max(r_S, r_O)$ .

We want to evaluate these equations for the case that the source is a distant star ( $r_S \rightarrow \infty$ ), the observer is on the Earth ( $r_O = 1$  AU), and the light ray is grazing the surface of the Sun ( $r_m = 0.0046$  AU). Then (67) gives the well-known deflection angle of

$$\Delta\varphi_0 = 1.75'' = 8.48 \times 10^{-6} \text{ rad} \quad (70)$$

and present day observations (see Will [29], Section 3.4) require

$$\Delta\varphi - \Delta\varphi_0 = \frac{1}{2} \left( -1.7 \pm 4.5 \right) \times 10^{-4} \Delta\varphi_0. \quad (71)$$

Comparison of (69) and (71) gives a bound for the possible values of  $\alpha(\tilde{r})$ ,

$$|\alpha(\tilde{r})| = 2 \left| \frac{\Delta\varphi - \Delta\varphi_0}{\Delta\varphi_0} \right| \left| \frac{\Delta\varphi_0}{\pi + \Delta\varphi_0} \right| \lesssim 2 \times 10^{-9}. \quad (72)$$

If we assume that the perturbation functions  $\phi_A(r)$  have a fall-off behaviour according to eq. (39), and if we neglect terms of second and higher order, the integral in (68) can be calculated numerically. For  $r_S \rightarrow \infty$ ,  $r_O = 1$  AU and  $r_m = 0.0046$  AU we find

$$\Delta\varphi - \Delta\varphi_0 = 4.2 \times 10^{-6} \phi_{11} + 5.1 \times 10^{-11} \phi_{21} - 4.2 \times 10^{-6} \phi_{01}. \quad (73)$$

We see that  $\phi_{21}$  contributes with a much smaller factor than the other two perturbations. This has its reason in the fact that in (61) the Finslerity  $\phi_2(r)$  comes with a factor that changes sign between  $r = r_m$  and  $r = r_O$ , therefore positive and negative contributions to the integral partly cancel out. Hence, light deflection is rather insensitive to the Finslerity of our spacetime model. Combining (73) with (71), and using (70) and (41), gives the quite insignificant bound

$$|8.2 \times 10^4 \phi_{11} + \phi_{21}| \lesssim 52. \quad (74)$$

## 7.2 Time delay of light rays

From (62) we find

$$dt = \left( 1 + \frac{1}{2} \beta(r) \right) \frac{dr}{\sqrt{B(r)}}. \quad (75)$$

Integration of this equation yields the travel time. The difference to the Newtonian travel time  $t_N$  is, by definition, the time delay  $\delta t$ ,

$$t_N + \delta t = \left( \int_{r_m}^{r_S} + \int_{r_m}^{r_O} \right) \left( 1 + \frac{1}{2} \beta(r) \right) \frac{dr}{\sqrt{B(r)}}. \quad (76)$$

In the unperturbed Schwarzschild spacetime, the time delay  $\delta t_0$  is given by

$$t_N + \delta t_0 = \left( \int_{r_m}^{r_S} + \int_{r_m}^{r_O} \right) \frac{dr}{\sqrt{B(r)}}, \quad (77)$$

hence

$$\delta t = \delta t_0 + \left( \int_{r_m}^{r_S} + \int_{r_m}^{r_O} \right) \frac{\beta(r) dr}{2 \sqrt{B(r)}}. \quad (78)$$

Using the mean value theorem, this can be rewritten as

$$\delta t = \delta t_0 + \frac{1}{2} \beta(\hat{r})(t_N + \delta t_0) \quad (79)$$

where  $\hat{r}$  is some radius value between  $r_m$  and  $\max(r_S, r_O)$ .

Time delays have been measured with radar signals since the 1960s. In the beginning Mars, Mercury and Venus were used as passive reflectors. In this case the round-trip travel time for a signal from the Earth to the planet and back is two times the one-way travel time (76) plus a correction taking the orbital motion of the Earth into account. Later time delay experiments used spacecraft. The most accurate experiment of this kind was made with radio signals sent to the Cassini spacecraft, with the result (see Will [29], Sec. 3.4) that

$$\frac{\delta t - \delta t_0}{\delta t_0} = (2.1 \pm 2.3) \times 10^{-5}. \quad (80)$$

The measurement was made when Cassini was at a distance of 8.43 AU from the Sun, and the distance of closest approach  $r_m$  was 1.6 Solar radii ( $= 0.0074$  AU). This corresponds to  $\delta t_0 \approx 273 \mu\text{s}$  and  $t_N \approx 4700$  s. Hence we find, with (79) and (80), a very small bound for a certain linear combination of the perturbation functions at some radius value  $\hat{r}$  between 0.0074 AU and 8.43 AU,

$$|\beta(\hat{r})| = 2 \left| \frac{\delta t - \delta t_0}{\delta t_0} \right| \left| \frac{\delta t_0}{t_N + \delta t_0} \right| \lesssim 5.2 \times 10^{-12}. \quad (81)$$

If only the leading-order terms in (39) are taken into account, numerical calculation of the integral in (78) yields (for  $r_m = 0.0074$  AU,  $r_S = 8.43$  AU and  $r_O = 1$  AU)

$$\delta t - \delta t_0 = (6.6 \times 10^{-5} \phi_{11} + 3.3 \times 10^{-6} \phi_{21} - 7.5 \times 10^{-5} \phi_{01}) \text{ s}. \quad (82)$$

The left-hand side can be estimated with the help of (80). With  $\delta t_0 \approx 273 \mu\text{s}$  and (41) we find

$$|20 \phi_{11} + \phi_{21}| \lesssim 3.6 \times 10^{-3}. \quad (83)$$

## 8 Effects on bound orbits

In Section 5 we have seen that circular orbits are affected only by the coefficient  $\phi_0$ , but not by  $\phi_1$  and  $\phi_2$ . In this section we consider non-circular bound orbits, and we will investigate how the perturbation functions influence Kepler's third law and the perihelion precession.

We consider a massive particle ( $2\mathcal{L} = -c^2$ ) on a bound orbit, with minimum radius  $r_1$  (perihelion) and maximum radius  $r_2$  (aphelion). We need to calculate how the constants of motion  $E$  and  $L$  depend on  $r_1$  and  $r_2$ . To that end, we rewrite (25) for the Schwarzschild metric coefficients, with  $2\mathcal{L} = -c^2$  and using again the abbreviation (56), in the following form.

$$\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4 (E^2 - c^4 r^2 p - L^2 c^2 p)}{c^2 L^2} \left( 1 - \phi_1 + 2\phi_2 - \frac{3\phi_2 L^2 c^2 p}{(E^2 - c^4 r^2 p)} \right) - \frac{\phi_0 E^2 r^4}{c^2 L^2}. \quad (84)$$

For  $r = r_1$  and  $r = r_2$  the right-hand side of (84) has to vanish,

$$E^2 - c^4 r_1^2 p(r_1) - L^2 c^2 p(r_1) + \phi_0(r_1) E^2 = 0, \quad E^2 - c^4 r_2^2 p(r_2) - L^2 c^2 p(r_2) + \phi_0(r_2) E^2 = 0. \quad (85)$$

Solving for  $E^2$  and  $L^2$  yields

$$E^2 = E_0^2 + E_0^2 \left( \frac{\phi_0(r_2) p(r_1) - \phi_0(r_1) p(r_2)}{p(r_1) - p(r_2)} \right), \quad (86)$$

$$L^2 = L_0^2 + \frac{E_0^2}{c^2} \left( \frac{\phi_0(r_2) - \phi_0(r_1)}{p(r_1) - p(r_2)} \right), \quad (87)$$

where

$$E_0^2 = \frac{c^4 p(r_1) p(r_2) (r_2^2 - r_1^2)}{p(r_1) - p(r_2)} \quad \text{and} \quad L_0^2 = \frac{c^2 (p(r_2) r_2^2 - p(r_1) r_1^2)}{p(r_1) - p(r_2)} \quad (88)$$

are the values of the constants of motion in the unperturbed Schwarzschild spacetime. Substitution of (86) and (87) into (84), and using the identities  $E_0^2 = L_0^2 c^2 p(r_1) + c^4 p(r_1) r_1^2 = L_0^2 c^2 p(r_2) + c^4 p(r_2) r_2^2$ , gives the orbit equation in terms of  $E_0^2$  and  $L_0^2$ ,

$$\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4 (E_0^2 - c^4 r^2 p - L_0^2 c^2 p)}{c^2 L_0^2} \left( 1 - \phi_1 + 2\phi_2 - \frac{3\phi_2 L_0^2 c^2 p}{(E_0^2 - c^4 r^2 p)} \right) - \frac{c^2 E_0^2 \left\{ (\phi_0(r_2) - \phi_0(r)) (p(r_1) r_1^2 - p(r) r^2) - (\phi_0(r_1) - \phi_0(r)) (p(r_2) r_2^2 - p(r) r^2) \right\}}{L_0^2 (p(r_1) - p(r_2)) (E_0^2 - c^4 r^2 p - L_0^2 c^2 p)} \right). \quad (89)$$

After substituting  $E_0^2$  and  $L_0^2$  from (88) we find

$$\left( \frac{dr}{d\varphi} \right)^2 = C(r) (1 - \gamma(r)) \quad (90)$$

where

$$C(r) = \frac{r^2 (r_2 - r) (r - r_1)}{r_1 r_2} \left( 1 - \frac{2GM}{c^2} \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r} \right) \right) \quad (91)$$

and

$$\begin{aligned} \gamma(r) = & \phi_1(r) - 2\phi_2(r) + \frac{3\phi_2(r) r_1 r_2 \left( 1 - \frac{2GM}{c^2 r} \right)}{r(r_1 + r_2 - r) \left( 1 - \frac{2GM(r_1^2 + r_2^2 + r_1 r_2 - r r_1 - r r_2)}{c^2 r_1 r_2 (r_1 + r_2 - r)} \right)} \\ & + \frac{c^2 r (r_1 + r_2) \left( 1 - \frac{2GM}{c^2 r_1} \right) \left( 1 - \frac{2GM}{c^2 r_2} \right)}{2GM(r_2 - r_1) \left\{ 1 - \frac{2GM}{c^2} \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r} \right) \right\}} \left( \frac{r_2 (\phi_0(r_2) - \phi_0(r))}{r_2 - r} - \frac{r_1 (\phi_0(r_1) - \phi_0(r))}{r_1 - r} \right). \end{aligned} \quad (92)$$

For the limiting case of a circular orbit,  $r = r_1 = r_2$ , we find

$$\gamma(r) = \phi_1(r) + \phi_2(r) + \frac{c^2 r^2 \left( 1 - \frac{2GM}{c^2 r} \right)^2}{GM \left( 1 - \frac{6GM}{c^2 r} \right)} \left( \phi_0'(r) + \frac{r}{2} \phi_0''(r) \right). \quad (93)$$

Analogously, substitution of (86) and (87) into (24) results in

$$\left( \frac{dr}{dt} \right)^2 = D(r) (1 - \delta(r)) \quad (94)$$

where

$$D(r) = \frac{2GM(r_2 - r)(r - r_1) \left( 1 - \frac{2GM}{c^2 r} \right)^2 \left( 1 - \frac{2GM}{c^2} \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r} \right) \right)}{r^2 (r_1 + r_2) \left( 1 - \frac{2GM}{c^2 r_1} \right) \left( 1 - \frac{2GM}{c^2 r_2} \right)} \quad (95)$$

and

$$\begin{aligned} \delta(r) = \gamma(r) - 2\phi_0(r) - \frac{c^2 r_1 r_2 (\phi_0(r_2) - \phi_0(r_1))}{2GM(r_2 - r_1)} + \frac{r_2 \phi_0(r_2) - r_1 \phi_0(r_1)}{r_2 - r_1} \\ + \frac{2\phi_2(r)(r_2 - r)^2(r - r_1)^2 \left(1 - \frac{2GM}{c^2} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r}\right)\right)^2}{r^2 \left(r_1 + r_2 - r - \frac{2GM}{c^2 r_1 r_2} (r_1^2 + r_2^2 + r_1 r_2 - r(r_1 + r_2))\right)^2}. \end{aligned} \quad (96)$$

For the limiting case of a circular orbit,  $r = r_1 = r_2$ , we find

$$\delta(r) - \gamma(r) = -\phi_0(r) - \frac{c^2 r^2}{2GM} \left(1 - \frac{2GM}{c^2 r}\right) \phi_0'(r). \quad (97)$$

Equations (93) and (97) can be used as valid approximations for orbits whose eccentricity is not too big, where  $r$  is any value between the perihelion and the aphelion.

### 8.1 Perihelion precession

From (90) and (94) we find

$$d\varphi = \left(1 + \frac{1}{2}\gamma(r)\right) \frac{dr}{\sqrt{C(r)}}, \quad (98)$$

$$dt = \left(1 + \frac{1}{2}\delta(r)\right) \frac{dr}{\sqrt{D(r)}}. \quad (99)$$

Integrating these two equations over the orbit from one perihelion transit to the next,

$$2\pi + \Delta\Phi = 2 \int_{r_1}^{r_2} \left(1 + \frac{1}{2}\gamma(r)\right) \frac{dr}{\sqrt{C(r)}}, \quad (100)$$

$$T = 2 \int_{r_1}^{r_2} \left(1 + \frac{1}{2}\delta(r)\right) \frac{dr}{\sqrt{D(r)}}, \quad (101)$$

gives the anomalistic period  $T$  (in terms of coordinate time  $t$ ) and the angular advance  $\Delta\Phi$  of the perihelion during this period. We denote the corresponding quantities in the unperturbed Schwarzschild spacetime by an index 0,

$$2\pi + \Delta\Phi_0 = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{C(r)}}, \quad (102)$$

$$T_0 = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{D(r)}}. \quad (103)$$

The precession rate of the perihelion, in radians per time, is  $\omega = \Delta\Phi/T$ . In our linearised setting  $\omega$  deviates from  $\omega_0 = \Delta\Phi_0/T_0$  according to

$$\frac{\omega - \omega_0}{\omega_0} = \frac{1}{\Delta\Phi_0} \int_{r_1}^{r_2} \frac{\gamma(r) dr}{\sqrt{C(r)}} - \frac{1}{T_0} \int_{r_1}^{r_2} \frac{\delta(r) dr}{\sqrt{D(r)}}. \quad (104)$$

With the mean-value theorem the last equation can be rewritten as

$$\frac{\omega - \omega_0}{\omega_0} = \frac{\gamma(\hat{r})}{2} \left(\frac{2\pi}{\Delta\Phi_0} + 1\right) - \frac{\delta(\hat{r})}{2}, \quad (105)$$



where  $\hat{r}$  and  $\tilde{r}$  are some radius values between  $r_1$  and  $r_2$ .

For Mercury, the precession rate  $\omega_0$  according to general relativity is well-known to be 43 arcseconds per century. This corresponds to a precession angle (in radians) per revolution of  $\Delta\Phi_0 = 0.502 \times 10^{-6}$ . Present day observations confirm that the general-relativistic value is true, with a possible relative error of  $10^{-3}$ , see Will [29], Section 3.5. Hence, (105) implies

$$|6.2 \times 10^6 \gamma(\hat{r}) - 0.5 \delta(\tilde{r})| \leq 10^{-3}, \quad (106)$$

which is a restrictive bound on  $\gamma(\hat{r})$  for  $\hat{r}$  on the Mercury orbit.

If we take only leading-order terms of (39) into account, the integrals in (104) can be numerically calculated. With the Mercury values ( $\Delta\Phi_0 = 0.502 \times 10^{-6}$ ,  $T_0 = 87.969$  d,  $r_1 = 46\,001\,200$  km and  $r_2 = 69\,816\,900$  km) we find

$$\frac{\omega - \omega_0}{\omega_0} = 3.3 \times 10^{-1} \phi_{11} + 3.1 \times 10^{-1} \phi_{21} + 2.5 \times 10^{-8} \phi_{01}. \quad (107)$$

The observational fact that the left-hand side is bounded by  $10^{-3}$ , together with (41), implies that

$$|\phi_{11} + 9.4 \times 10^{-1} \phi_{21}| \lesssim 3.0 \times 10^{-3}. \quad (108)$$

In combination with (83) this gives us a bound for  $\phi_{21}$ ,

$$|\phi_{21}| \lesssim 3.6 \times 10^{-3}, \quad (109)$$

which means that the Finslerity is bounded by

$$|\phi_2(r)| \lesssim 1.8 \times 10^{-10} \quad (110)$$

everywhere in the Solar system beyond the Mercury orbit.

## 9 Conclusions

In this paper we have considered a class of spherically symmetric and static Finsler spacetimes which are small perturbations of the Schwarzschild metric. After fixing the ambiguity in the choice of the radial coordinate by requiring that a sphere at coordinate  $r$  has area  $4\pi r^2$ , the perturbed metric is characterised by three functions  $\phi_0(r)$ ,  $\phi_1(r)$  and  $\phi_2(r)$  which we called the “perturbation functions”. It was our main goal to determine the bounds which are imposed on these perturbation functions by observations in the Solar system. In this way we have provided a framework for testing if a certain Finsler modification of general relativity is in agreement with experimental facts.

We have been careful to set up the formalism in such a way that not only freely falling particles but also light rays are unambiguously defined as Finsler geodesics. We feel that this is a major advantage in comparison to several other Finsler approaches where the definition of light rays is questionable. Having both freely falling particles and light rays at our disposal is essential because these are the tools needed for the experiments discussed.

The formalism presented here is meant as an analogue of the PPN formalism. From a methodological point of view, there are two differences. First, our formalism is post-Schwarzschild rather than post-Newtonian. This is, of course, motivated by the fact that we wanted to concentrate on the possible Finsler deviations from standard general relativity. Second, we chose for the radial coordinate the area coordinate, whereas in the standard PPN formalism one chooses the isotropic radial coordinate. This is a necessary deviation from the standard PPN formalism because the isotropic radial coordinate does not exist in a proper Finsler spacetime.

Our approach is purely kinematical, i.e., no field equation is used. Hence, one can use it for testing the validity of solutions to *any* Finslerian field equation, provided that the solutions belong to the class considered in this paper.

Here we have discussed only tests where all objects moving in the field of the Sun can be treated as test particles. One could set up a Finsler geometry model for more complicated situations, e.g. for the motion of the Earth in the combined gravitational field of the Sun and the Moon. This would make more sensitive tests possible, in particular using the very precise Lunar Laser Ranging measurements. This is planned to be done in future work.

## Appendix

For our work it was crucial that the timelike Finsler geodesics were interpreted as freely falling particles and the lightlike geodesics were interpreted as light rays. We feel that this is the most natural interpretation, if one believes in the possibility that our real world carries a Finsler spacetime structure. As far as the light rays are concerned, the geodesic hypothesis can be further justified by deriving the lightlike geodesics as the bicharacteristic curves of appropriately generalised Maxwell equations. In this appendix we will outline how this can be done. As a detailed treatment would require a separate paper, we will only sketch the line of thought, just enough to convince the reader that the geodesic hypothesis for light rays is, indeed, well motivated.

In Definition 1 we have defined Finsler spacetimes in terms of a Lagrangian  $\mathcal{L}(x, \dot{x})$ . We can switch to a Hamiltonian formulation by introducing canonical momenta

$$p_\mu = \frac{\partial \mathcal{L}(x, \dot{x})}{\partial \dot{x}^\mu} = g_{\mu\nu}(x, \dot{x}) \dot{x}^\nu \quad (111)$$

and the Hamiltonian

$$\mathcal{H}(x, p) = p_\mu \dot{x}^\mu - \mathcal{L}(x, \dot{x}) = \frac{1}{2} g^{\mu\nu}(x, p) p_\mu p_\nu \quad (112)$$

where  $g^{\mu\nu}(x, p)$  is defined through

$$g^{\mu\nu}(x, p) g_{\nu\sigma}(x, \dot{x}) = \delta_\sigma^\mu. \quad (113)$$

In (112) and (113),  $\dot{x}^\mu$  must be expressed as a function of  $x$  and  $p$  with the help of (111). For later convenience, we write

$$\mathcal{H}^\mu(x, p) = \frac{\partial \mathcal{H}(x, p)}{\partial p_\mu}. \quad (114)$$

Note that, because  $\mathcal{L}(x, \dot{x})$  is assumed to be homogeneous of degree two with respect to the  $\dot{x}^\mu$ , the Hamiltonian  $\mathcal{H}(x, p)$  is homogeneous of degree two with respect to the  $p_\mu$ , hence

$$p_\mu \mathcal{H}^\mu(x, p) = 2 \mathcal{H}(x, p). \quad (115)$$

The lightlike Finsler geodesics are the solutions to Hamilton's equations with  $\mathcal{H}(x, p) = 0$ . It is our goal to demonstrate that these are the bicharacteristic curves of appropriately generalised Maxwell equations.

In the case of a pseudo-Riemannian metric, where the  $g^{\mu\nu}$  depend on  $x$  only and not on  $p$ , the source-free vacuum Maxwell equations can be written as

$$\partial_\mu F_{\nu\sigma}(x) + \partial_\nu F_{\sigma\mu}(x) + \partial_\sigma F_{\mu\nu}(x) = 0 \quad (116)$$

$$(\mathcal{H}^\mu(x, \partial) + \dots) F_{\mu\nu} = 0. \quad (117)$$

Here  $F_{\mu\nu}$  is the electromagnetic field tensor and  $\partial_\mu$  means partial derivative with respect to  $x^\mu$ . In (117) we have written only the principal part.  $\mathcal{H}^\mu(x, \partial) = g^{\mu\sigma}(x) \partial_\sigma$  is a first-order differential operator and, thus, homogeneous of degree one with respect to the  $\partial_\sigma$ . The omitted terms, indicated by ellipses, involve the Christoffel symbols and no derivatives, so they are in particular homogeneous of degree zero with respect to the  $\partial_\mu$ .

It is very natural to postulate that Maxwell's equations take the same form of (116) and (117) on a Finsler spacetime. Now  $\mathcal{H}^\mu(x, p)$  is no longer a polynomial, but still homogeneous of degree one with respect to the  $p_\mu$ , so  $\mathcal{H}^\mu(x, \partial)$  is no longer a differential operator but still a well-defined pseudo-differential operator. (For a detailed exposition of pseudo-differential operators see, e.g., Hörmander [11].) The terms indicated by ellipses are necessary in (117) to make this equation coordinate-independent. Their special form, however, will not be relevant for the following argument. We only have to assume that they are homogeneous of degree one with respect to the  $\partial_\mu$ , as in the pseudo-Riemannian case.

We now apply the operator  $\mathcal{H}^\mu(x, \partial)$  to (116). With the help of (117) and (115) we find

$$\mathcal{H}(x, \partial) F_{\nu\sigma} + \dots = 0 \quad (118)$$

where again only the principal part (i.e., the terms of highest degree of homogeneity) has been written out. This demonstrates that the field tensor satisfies a (pseudo-differential) Finslerian wave equation. The principal part determines the *characteristic equation* (or *eikonal equation*)

$$\mathcal{H}(x, \partial S) = 0. \quad (119)$$

It gives the characteristic surfaces  $S = \text{const}$  along which solutions  $F_{\mu\nu}(x)$  to Maxwell's equations might have discontinuities of their first derivatives. The characteristic equation has the form of a Hamilton-Jacobi equation with the Finsler Hamiltonian  $\mathcal{H}(x, p)$ . The *bicharacteristic curves* (or *rays*) are the corresponding solutions to Hamilton's equations, i.e., the lightlike Finsler geodesics.

We have thus derived the result that light rays are lightlike Finsler geodesics from the assumption that Maxwell's equations on a Finsler spacetime take the form of (116) and (117). A full treatment of the subject would, of course, require to specify the omitted terms in (117) and to discuss their implications for physics. This could be the subject of another paper.

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## References

- [1] A. K. Aringazin and G. S. Asanov. Finslerian post-Riemannian corrections to the equations of geodesics. *Gen. Relativ. Gravit.*, 17:1153–1163, 1985.
- [2] G. S. Asanov. *Finsler geometry, relativity and gauge theories*. Reidel, Dordrecht, 1985.
- [3] G. S. Asanov. Finslerian extension of Schwarzschild metric. *Fortschritte der Physik*, 40:667–693, 1992.
- [4] J. K. Beem. Indefinite Finsler spaces and timelike spaces. *Can. J. Math.*, 22:1035–1039, 1970.
- [5] G. Y. Bogoslovsky. On a special-relativistic theory of anisotropic space-time. *Dokl. Akad. Nauk SSSR*, 213:1055–1058, 1973.
- [6] Z. Chang and X. Li. Modified Newton's gravity in Finsler space as a possible alternative to dark matter hypothesis. *Phys. Lett.*, B 668:453–456, 2009.
- [7] A. A. Coley. Clocks and gravity. *Gen. Relativ. Gravit.*, 14:1107–1114, 1982.
- [8] F. Francisco, O. Bertolami, P. J. S. Gil, and J. Páramos. Modelling the reflective thermal contribution to the acceleration of the Pioneer spacecraft. *Phys. Lett.*, B 711:337–346, 2012.
- [9] G. Gibbons, J. Gomis, and C. N. Pope. General very special relativity is Finsler geometry. *Phys. Rev.*, D 76:081701, 2007.
- [10] F. Girelli, S. Liberati, and L. Sindoni. Planck-scale modified dispersion relations and Finsler geometry. *Phys. Rev.*, D 75:064015, 2007.
- [11] L. Hörmander. *The analysis of linear partial differential operators* III, IV. Springer, Heidelberg, 1985.
- [12] M. S. Knebelman. Collineations and motions in generalized spaces. *Amer. J. Math.*, 51:527–564, 1929.

- [13] V. A. Kostelecký. Riemann-Finsler geometry and Lorentz-violating kinematics. *Phys. Lett.*, B 701:137–143, 2011.
- [14] C. Lämmerzahl, D. Lorek, and H. Dittus. Confronting Finsler space-time with experiment. *Gen. Relativ. Gravit.*, 41:1345–1353, 2009.
- [15] X. Li and Z. Chang. A possible scenario of the Pioneer anomaly in the framework of Finsler geometry. *Phys. Lett.*, B 692:1–3, 2010.
- [16] P. J. McCarthy and S. F. Rutz. The general four-dimensional spherically symmetric Finsler space. *Gen. Relativ. Gravit.*, 25:589–602, 1993.
- [17] C. Pfeifer and M. N. R. Wohlfarth. Causal structure and electrodynamics on Finsler spacetimes. *Phys. Rev.*, D 84:044039, 2011.
- [18] C. Pfeifer and M. N. R. Wohlfarth. Finsler geometric extension of Einstein gravity. *Phys. Rev.*, D 85:064009, 2012.
- [19] E. V. Pitjeva. High-precision ephemerides of planets – EPM and determination of some astronomical constants. *Solar System Research*, 39:176–186, 2005.
- [20] B. Rievers and C. Lämmerzahl. High precision thermal modeling of complex systems with application to the flyby and Pioneer anomaly. *Ann. Phys. (Leipzig)*, 523:439–449, 2011.
- [21] I. W. Roxburgh. Post-Newtonian limit of Finsler space theories of gravity and solar system tests. *Gen. Relativ. Gravit.*, 24:419–431, 1992.
- [22] I. W. Roxburgh and R. K. Tavakol. Non-Riemann geometrizable effects in the gravitational one-body problem. *Gen. Relativ. Gravit.*, 10:307–312, 1979.
- [23] H. Rund. *The differential geometry of Finsler spaces*. Springer, Berlin, 1959.
- [24] H. Rund and J. Beare. *Variational properties of direction-dependent metric fields*. University of South Africa, Cape Town, 1972.
- [25] S. Rutz. Symmetry in Finsler spaces. In D. Bao, S.-S. Chern, and Z. Shen, editors, *Finsler geometry*, volume 196 of *Contemporary Mathematics*, pages 289–299, Providence, Rhode Island, USA, 1996. American Mathematical Society.
- [26] S. Rutz. A Finsler generalisation of Einstein’s vacuum field equations. *Gen. Relativ. Gravit.*, 25:1139–1153, 1998.
- [27] L. I. Schiff. On experimental tests of the general theory of relativity. *Amer. J. Phys.*, 28:340–343, 1960.
- [28] S. G. Turyshev, V. T. Toth, G. Kinsella, S.-C. Lee, S. M. Lok, and J. Ellis. Support for the thermal origin of the Pioneer anomaly. *Phys. Rev. Lett.*, 108:241101, 2012.
- [29] C. M. Will. The confrontation between general relativity and experiment. *Living Rev. Relativity*, 9(3), 2006. <http://www.livingreviews.org/lrr-2006-3>.